Euler's Method: Numerically approximating solutions of $dy/dt = f(t,y)$

1) Choose step size $\Delta t > 0$
2) Compute $f(t_0, y_0)$, the slope of the solution of the IVP at time t_0
3) Construct line w/ that tangent --

$y_1 = y_0 + \Delta t f(t_0, y_0)$, $t_1 = t_0 + \Delta t$

So, $y_1 \approx y(t_1)$

4) Repeat! Compute $f(t_1, y_1)$ $\quad y_{n+1} = y_n + \Delta t f(t_n, y_n)$

$y_2 = y_1 + \Delta t f(t_1, y_1)$ $\quad t_{n+1} = t_n + \Delta t$

$t_2 = t_1 + \Delta t$

Looking for error in Euler's method

- tangent line is first degree Taylor polynomial
- So $|y_1 - y(t_1)| \le \frac{k_2}{2!}\Delta t^2$ $\quad k_2 = \max \frac{d^2y}{dt^2}$ OR $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial y}$ ← b/c $y'' = f'$ $\left(f = \frac{dy}{dt}\right)$
- each step has max error $k\Delta t^2$ (tangent line instead of solution)
- 2nd step: computing slope @ approximate point

$|f(t_1, y_1) - f(t, y(t))| \le \left|\frac{\partial F}{\partial y}\right| \cdot k\Delta t^2$

- each step adds to error of size $k\Delta t^2$

How many steps needed to approximate $y(t_0+1)$? $1/\Delta t$ steps

So, error at t_0+1 bounded by: ← constant

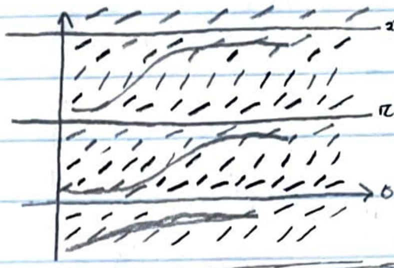
Error per step * # of steps: $k\Delta t^2 * \frac{1}{\Delta t} = K\Delta t$ ← you pick!

Qualitative Analysis

$\frac{dy}{dt} = e^{y^{3/10}} \sin^2 y$

$\int \frac{dy}{e^{y^{3/10}}\sin^2 y}$ can't be evaluated, so use qualitative methods

- $e^{y^{3/10}}\sin^2 y$ is always positive unless $y = n\pi$, these lines are equilibria solutions
- between these lines, solutions must always increase



Phase lines

Logistic: $\frac{dP}{dt} = kP(1 - P/C)(P - B)$ if $P < B$, $\frac{dP}{dt} < 0$

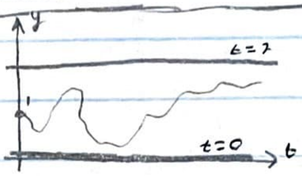
So, for $B < P(0) < C$, population increases as $t \to \infty$

For $P(0) > C$, population decreases. $P(t) \to C$

For $0 < P(0) < B$, population decreases $P(t) \to 0$ as $t \to \infty$

Using the uniqueness theorem

$\frac{dy}{dt} = y(\cos(t^2 + y^2)e^y + y^{14} + \frac{y}{t^2+17})(y-2)$, $y(0) = 1$



equilibrium solutions: $y(t) = 0$, $y(t) = 2$

By the uniqueness theorem, if $0 < y(0) < 2$, then for all t $0 < y(t) < 2$ for solution

One more linear diff eq

$\frac{dy}{dt} = -2y + 5e^{-2t}$

$y_g(t) = ke^{-2t}$

Guess $y_p(t) = \alpha t e^{-2t}$

$\frac{d(\alpha t e^{-2t})}{dt} = 2(\alpha t e^{-2t}) + 5e^{-2t}$

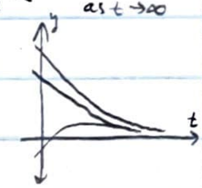
$\alpha e^{-2t} - 2\alpha t e^{-2t} = 2\alpha t e^{-t} + 5e^{-2t}$

$\alpha e^{-2t} = 5e^{-2t} \to \alpha = 5$

$y_p(t) = 5te^{-2t}$

$y(t) = ke^{-2t} + 5te^{-2t}$

slower: $ke^{-2t} \to 0$, $5te^{-2t} \to 0$ as $t \to \infty$



Linear differential Equations

$\frac{dy}{dt} = -3y + e^{-t}$ $\begin{pmatrix} a(t) = -3 \\ b(t) = e^{-t} \end{pmatrix}$

$y_g(t) = ke^{-3t}$

Guess $y_p(t) = \alpha e^{-t}$

check:

$\frac{d(\alpha e^{-t})}{dt} = -3(\alpha e^{-t}) + e^{-t}$

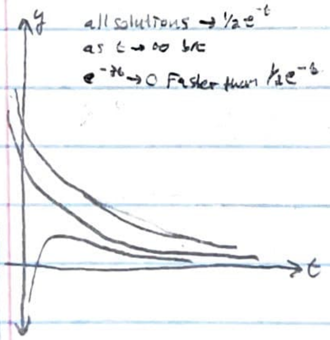
$-\alpha e^{-t} = -3\alpha e^{-t} + e^{-t}$

$\to -\alpha = -3\alpha + 1$

$\alpha = 1/2$, so $y_p(t) = \frac{1}{2}e^{-t}$

$y(t) = ke^{-3t} + 1/2 e^{-t}$

all solutions $\to 1/2 e^{-t}$ as $t \to \infty$ b/c $e^{-3t} \to 0$ faster than $1/2 e^{-t}$



$\frac{dy}{dt} = -2y + 3t$

$y_g(t) = ke^{-2t}$

$y_p(t)$ guess: $y_p(t) = \alpha t$

$\frac{d\alpha t}{dt} = -2(\alpha t) + 3t$

$\alpha = (-2\alpha + 3)t$ ← no α makes this true for all t

guess: $y_p(t) = \alpha t + \beta$

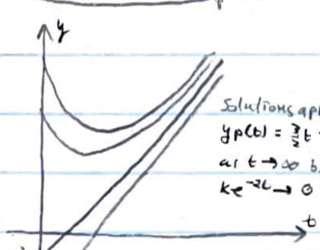
$\frac{d(\alpha t + \beta)}{dt} = -2(\alpha t + \beta) + 3t$

$\alpha = (-2\alpha + 3)t - 2\beta$, needs to be = for all t so

$\alpha = -2\beta$, $-2\alpha + 3 = 0$

$\alpha = 3/2$, $\beta = -3/4$, $y_p(t) = \frac{3t}{2} - 3/4$

$y(t) = ke^{-2t} + \frac{3t}{2} - \frac{3}{4}$



Solutions approach $y_p(t) = \frac{3}{2}t - 3/4$ as $t \to \infty$ b/c $ke^{-2t} \to 0$ very fast

$\frac{dy}{dt} = -2y + 7\cos(3t)$

$y_g(t) = ke^{-2t}$

guess: $y_p(t) = \alpha\cos(3t) + \beta\sin(3t)$

$\frac{d(\alpha\cos(3t) + \beta\sin(3t))}{dt} = -2(\alpha\cos(3t) + \beta\sin(3t)) + 7\cos 3t$

$-3\alpha\sin(3t) + 3\beta\sin(3t) = -2\beta\sin(3t) + (7 - 2\alpha)\cos(3t)$

$-3\alpha = 2\beta$ $\quad 7 - 2\alpha = 3\beta$

$\alpha = \frac{2\beta}{3} \to 7 - 2\left(\frac{2\beta}{3}\right) = 3\beta$

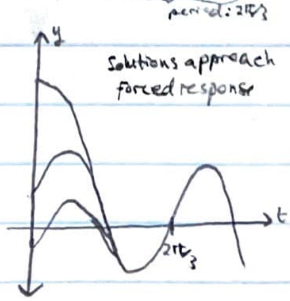
$21 - 4\beta = 9\beta$

$\beta = \frac{21}{13} \to \alpha = \frac{2(\frac{21}{13})}{3} \to \alpha = \frac{42}{39} \to \alpha = \frac{14}{13}$

$y_p(t) = \frac{14}{13}\cos 3t + \frac{21}{13}\sin 3t$

$y(t) = ke^{-2t} + \underbrace{\frac{14}{13}\cos 3t + \frac{21}{13}\sin 3t}_{\text{period: } 2\pi/3}$

Solutions approach forced response



Simple Computations w/ Linear Equations

$\frac{dy}{dt} = -3y + 5\cos(8t)$ → what can we say about solutions using simple calculations

Natural response $y_g(t) = ke^{-3t} \to 0$ fast as $t \to \infty$, so all solutions → forced response as $t \to \infty$

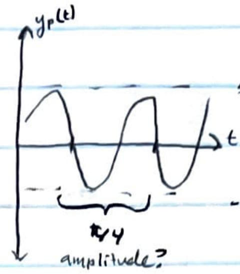
Forced response would be

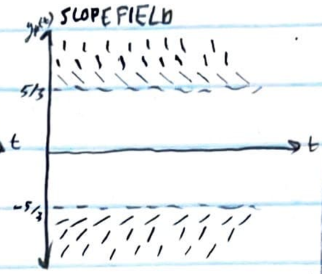
Guess $y_p(t) = \alpha\cos(8t) + \beta\sin(8t)$

know forced response is a periodic function w period $\pi/4$

Slope field for $\frac{dy}{dt} = -3y + 5\cos(8t)$

$-5 < 5\cos(8t) < 5$ for all t





So, when y is big, $\frac{dy}{dt} < 0$

when $3y > 5 \to y > 5/3$, $dy/dt < 0$

This means all solutions decrease if $y > 5/3$ and increase if $y < -5/3$

So, all solutions enter $-5/3 \leq y \leq 5/3$ and never get out → force response is in this strip

Amplitude of y_p is constrained $-5/3 \leq y_p(t) \leq 5/3$ for all t

More qualitative analysis of linear differential equations

Sometimes don't know $b(t)$ (external signal), but can use minimal info on $b(t)$ to get info on solutions

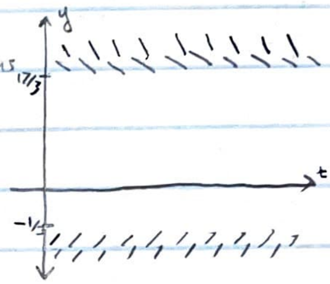
$dy/dt = -3y + b(t)$, $-1 \leq b(t) \leq 17$ for all t

$y_g(t) = ke^{-3t}$ natural response → 0 as $t \to \infty$ so all solutions → forced response as $t \to \infty$

$3y > 17$ or $y > \frac{17}{3} \Rightarrow \frac{dy}{dt} = -3y + b(t) < 0$ regardless of t

$3y < -1$ or $y < -1/3 \Rightarrow dy/dt > 0$ for all t

So, forced response is in interval $-1/3 \leq y \leq 17/3$, so all solutions enter this strip & never leave



Harmonic Oscillator

friction proportional to v^2

- mass m on spring w/ damper
- model motion: let $y(t)$ = position at t, $y = 0$ = rest position

So, $m\frac{d^2y}{dt^2}$ = Force = spring + damper

Hooke's Law: $F = -ky$, $k > 0$ constant (Force ∝ displacement) for small y

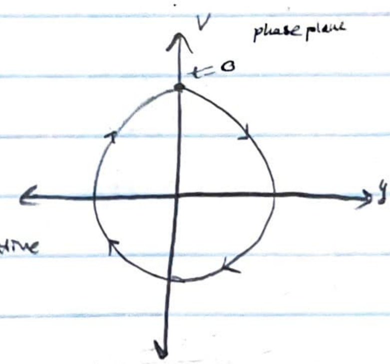
Spring-starting force (toward $y = 0$)

$$m\frac{d^2y}{dt^2} = -ky$$

Force of damper: (not friction) tries to slow you down, ∝ to $-\frac{dy}{dt}$

$$F_d = -b\frac{dy}{dt}$$

So, $m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$ $\xrightarrow{\text{OR}}$ $\frac{d^2y}{dt^2} = \frac{-k}{m}y - \frac{b}{m}\frac{dy}{dt}$ $\xrightarrow{\text{OR}}$ $\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$



Introduce new dependent variable $v(t) = dy/dt$ velocity and use it to get rid of second derivative

$\frac{dy}{dt} = v$

$\frac{dv}{dt} = \frac{-k}{m}y - \frac{b}{m}v$

} 1st order autonomous system

Simplest case: $b = 0$, $k = m = 1$ (undamped)

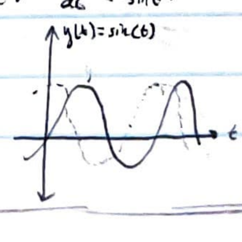
$dy/dt = v$

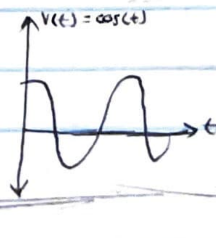
$dv/dt = -y$

guess $y(t) = \sin t$, $v(t) = \cos t$

$\frac{d\sin t}{dt} = \cos t$ ✓ $\frac{d\cos t}{dt} = -\sin t$ ✓

helix!



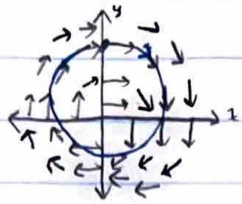


Qualitative Techniques for 2d Autonomous Systems

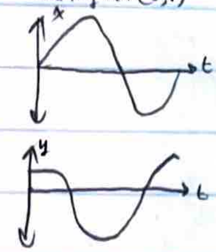
e.g. $F(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix} = \langle y, -x \rangle$

$F(1,0) = \langle 0,-1 \rangle$
$F(2,0) = \langle 0,-2 \rangle$
$F(1,1) = \langle 1,-1 \rangle$
$F(0,1) = \langle 1,0 \rangle$
$F(1, 1/2) = \langle 1/2, -1 \rangle$

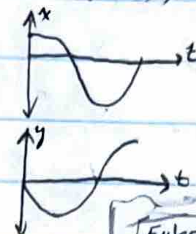
DIRECTION FIELD:



Starting at (0,1)



Starting at (1,0)



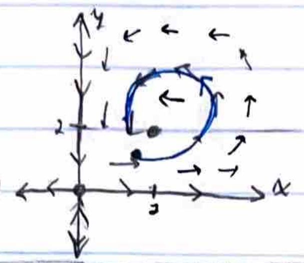
Steps:

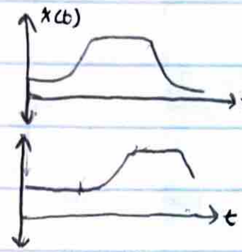
1) Find equilibria by solving $F(Y) = \vec{0} = \begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$
2) Draw direction field on xy plane
3) Sketch solutions on (x,y) plane
4) Rough sketch of t vs x(t), graphs t vs y(t),

e.g. $\begin{cases} dx/dt = 2x - xy \\ dy/dt = -y + \frac{1}{2}xy \end{cases}$

equilibria: $x=0, y=0$
$x=2, y=2$

$0 = 2x - xy$ $\quad$ $0 = -y + \frac{1}{2}xy$
$y = 2$ $\quad$ $x = 2$





Eulers method for Systems

Given $\begin{cases} dY/dt = F(Y) \\ Y(0) = Y_0 \end{cases}$

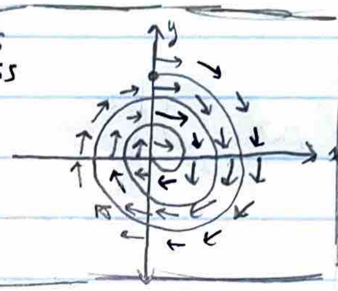
1) Pick step size $\Delta t > 0$
2) Compute $F(Y_0)$
3) $t_1 = 0 + \Delta t$, $Y_1 = Y_0 + \Delta t F(Y_0)$
4) Compute $F(Y_1)$
5) $t_2 = t_1 + \Delta t$, $Y_2 = Y_1 + \Delta t F(Y_1)$
6) REPEAT

Using Uniqueness in 2d systems

$\begin{cases} dx/dt = y \\ dy/dt = -x - 0.2y \end{cases}$ equilibrium at (0,0)

SOLUTIONS CANT CROSS

Creates a spiral toward 0, so all solutions "inside" are trapped and $\to 0$ as $t \to \infty$



SIR Model (Infectious disease)

S = susceptibles: haven't had disease
I = infected: have disease & can spread
R = removed: have had disease, not spreading (dead or immune)

- You can only get disease once
- susceptibles catch disease at rate jointly proportional to S & I
- Infecteds become removed at rate ∝ to I

$S(t)$ = fraction susceptible
$I(t)$ = fraction infected
$R(t)$ = fraction removed

So, $S(t) + I(t) + R(t) = 1$ for all t,
$R(t) = 1 - S(t) + R(t)$ so only need to know $(S(t), R(t))$

$\begin{cases} dS/dt = -\alpha S I \\ dI/dt = \alpha S I - \beta I \end{cases}$

$\left(\frac{dR}{dt} = \beta I\right)$
~~write~~ the second to 0 b/c total pop = constant)

α: how well mixed is the population? how often do they wash their hands?
β = how much room in the hospital? do you stay home when you're sick?

Find equilibria:

$\begin{cases} -\alpha S I = 0 \to S = 0 \text{ OR } I = 0 \\ \alpha S I - \beta I = 0 \to I(\alpha S - \beta) = 0 \quad I = 0, \; S = \beta/\alpha \end{cases}$

So, equilibria are all points where $I = 0 \to (S, 0)$ for any S

Draw direction field!! dS/dt always decreasing in region of interest
$dI/dt = I(\alpha S - \beta)$, $dI/dt > 0$ if $S > \beta/\alpha$

Initial condition: I close to 0, S close to 1, R = 0

For $\alpha = .2$, $\beta = .1$, $S(0) = .99$, $I(0) = .01$

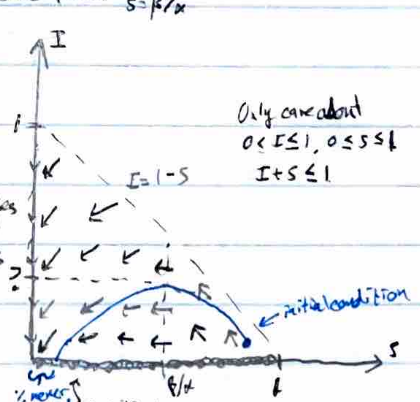
$I_{max}(t)$ at $t = 30$, ~.16

S final: ~20%

a 5% improvement of α and 10% improvement ↑ of β decreases max % infected & total % infected

Vaccines decreases % susceptibles at t=0 (turns them to R)
- get S_0 to left of β/α and avoid epidemic!!

Only care about
$0 < I \leq 1$, $0 \leq S \leq 1$
$I + S \leq 1$



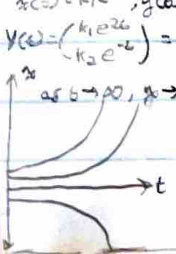
Simple, Solveable systems

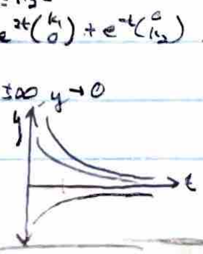
$\begin{cases} dx/dt = 2x \\ dy/dt = -y \end{cases}$ DECOUPLED

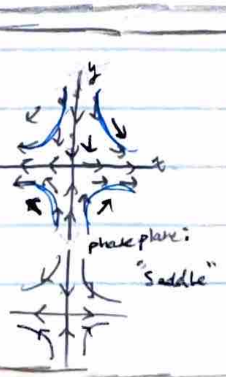
$x(t) = k_1 e^{2t}$, $y(t) = k_2 e^{-t}$

$Y(t) = \begin{pmatrix} k_1 e^{2t} \\ k_2 e^{-t} \end{pmatrix} = e^{2t}\begin{pmatrix} k_1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ k_2 \end{pmatrix}$

as $t \to \infty$, $x \to \pm\infty$, $y \to 0$







phase plane: "Saddle"

$\begin{cases} dx/dt = x + 5y \\ dy/dt = -2y \end{cases}$

equilibria: $x + 5y = 0 \to 0,0$
$-2y = 0$

$dy/dt = -2y$, $y(t) = k_2 e^{-2t}$

So, $dx/dt = x + 5k_2 e^{-2t}$

$x_h(t) = k_1 e^t$

$x_p(t) = \alpha e^{-2t}$

$\frac{d\alpha e^{-2t}}{dt} = \alpha e^{-2t} + 5k_2 e^{-2t}$

$-2\alpha = \alpha + 5k_2$

$\alpha = -5/3\, k_2$

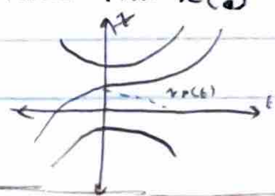
$x(t) = k_1 e^t - \frac{5}{3} k_2 e^{-2t}$

$Y(t) = k_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} -5/3 \\ 1 \end{pmatrix}$

IP $Y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -5/3 \\ 1 \end{pmatrix}$

$2 = k_1 - 5/3\, k_2$ $\quad$ $\boxed{k_2 = 3, \; k_1 = 7}$
$3 = k_2$

Particular: $Y(t) = 7e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3e^{-2t}\begin{pmatrix} -5/3 \\ 1 \end{pmatrix}$



One more simple, solvable system

$$\begin{cases} dx/dt = -3x \\ dy/dt = 4x - 3y \end{cases}$$

equilibria: $0 = -3x$ → $(0,0)$
$0 = 4x - 3y$

$dx/dt = -3x \rightarrow x(t) = k_1 e^{-3t}$

$dy/dt = 4k_1 e^{-3t} - 3y$

$y_g(t) = k_2 e^{-3t}$

$y_p(t) = \alpha t e^{-3t}$

$\alpha e^{-3t} - 3\alpha t e^{-3t} = -3\alpha t e^{-3t} + 4k_1 e^{-3t}$

$\alpha = 4k_1 \rightarrow y_p(t) = 4k_1 t e^{-3t}$

$$Y(t) = e^{-3t}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ 4k_1 \end{pmatrix}$$

As $t \to \infty$, $y \to 0$.

1. a) Say everything you can about: $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + 5y = 0$

1. Characteristic polynomial
2. Eigenvalues
3. Classify, describe behavior of solutions
4. Natural period: $2\pi/\beta$
5. General solution
6. Graph

General solution of an underdamped harmonic oscillator w/ $\lambda = \alpha \pm \beta i$

$y(t) = k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t)$ (natural period $2\pi/\beta$)

General solution of a critically damped harmonic oscillator:

$y(t) = k_1 e^{-p/2 t} + k_2 t e^{-p/2 t}$

General solution of an overdamped harmonic oscillator:

$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$

General solution of an undamped harmonic oscillator:

$y(t) = k_1 \cos(\beta t) + k_2 \sin(\beta t)$ period: $2\pi/\sqrt{q}$

2. b) Say everything you can about: $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + 5y = g(t)$

1. describe forcing
2. describe behavior of natural response as $t \to \infty$ & behavior of solutions based on this
3. Find period of forced response and compare to period of natural response
4. Find general solution of unforced & forced equations
5. Find amplitude of forced equation
6. Graph!

Diff eq Midterm #2 Study Guide

Determinants for linear systems:
- $\det(A) \neq 0$, only $(0,0)$ is an equilibrium
- $\det(A) = 0$, $A \neq \begin{bmatrix}0&0\\0&0\end{bmatrix}$, one line of equilibrium through origin
- $\det(A) = 0$, $\begin{bmatrix}0&0\\0&0\end{bmatrix} = A$, all points equilibrium

2D Autonomous Linear Systems

Form: $\frac{dY}{dt} = AY = \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \rightarrow \begin{cases}\frac{dx}{dt} = ax+by\\ \frac{dy}{dt} = cx+dy\end{cases}$

Steps to say everything you can about solutions of a linear equation:

1. Find characteristic polynomial of $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$(a-\lambda)(d-\lambda) - bc = \lambda^2 - \underbrace{(a+d)}_{\text{trace of } A}\lambda + \underbrace{(ad-bc)}_{\det A}$$

2. Find roots which are eigenvalues λ_1, λ_2
3. Classify the system & say long term behavior of solutions
4. Find eigenvector Y_1 for λ_1, Y_2 for λ_2 and write the general solution $Y(t) = k_1 e^{\lambda_1 t} Y_1 + k_2 e^{\lambda_2 t} Y_2$

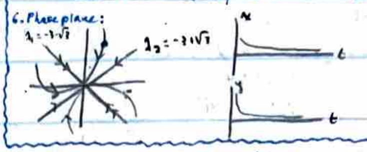
e.g. Procedure for cases 1, 2, 3

$\frac{dY}{dt} = \begin{bmatrix}-2&1\\2&-4\end{bmatrix}Y$ $\det(A) = (-2)(-4) - (1)(2) = 6$; $6 \neq 0$, $\vec{0}$ is the only equilib.

1. Char poly: $(-2-\lambda)(-4-\lambda) - 2 = \lambda^2 + 6\lambda + 6 = 0$
2. Roots: $\lambda = \frac{-6 \pm \sqrt{36-24}}{2}$ $\lambda_1 = -3-\sqrt{3}$, $\lambda_2 = -3+\sqrt{3}$ $\lambda_1 < \lambda_2 < 0$, SINK
3. Class: since $\lambda_1 < \lambda_2 < 0$, system is a sink, solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$
4. Eigenvectors: $\lambda_1 = -3-\sqrt{3}$: $-2x+y = (-3-\sqrt{3})x$, $y = (-1-\sqrt{3})x$, So $\begin{pmatrix}1\\-1-\sqrt{3}\end{pmatrix}$; $\lambda_2 = -3+\sqrt{3}$: $-2x+y = (-3+\sqrt{3})x$, $y = (-1+\sqrt{3})x$, So $\begin{pmatrix}1\\-1+\sqrt{3}\end{pmatrix}$
5. General solution:

$$Y(t) = k_1 e^{(-3-\sqrt{3})t}\begin{pmatrix}1\\-1-\sqrt{3}\end{pmatrix} + k_2 e^{(-3+\sqrt{3})t}\begin{pmatrix}1\\-1+\sqrt{3}\end{pmatrix}$$

6. Phase plane:



e.g. Procedure for cases 4, 5

$\frac{dY}{dt} = \begin{bmatrix}-1&5\\-2&-1\end{bmatrix}Y$ $\det(A) = (-1)(-1) - 5(-2) = 11$; $11 \neq 0$, $\vec{0}$ is the only equilib.

1. Char poly: $(-1-\lambda)(-1-\lambda) - (5)(-2) = 0$; $\lambda^2 + 2\lambda + 11 = 0$
2. Roots: $\lambda = \frac{-2 \pm \sqrt{4-44}}{2}$ $\lambda_1 = -1+i\sqrt{10}$, $\lambda_2 = -1-i\sqrt{10}$
3. Class: $\alpha < 0$, so spiral sink (solutions oscillate $\rightarrow \vec{0}$ as $t \rightarrow \infty$)
4. Period: $T = 2\pi/\beta$, so period is $2\pi/\sqrt{10}$
5. Direction of spiral? Plug in a point: $\begin{bmatrix}-1&5\\-2&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$ solutions must spiral clockwise
6. Find eigenvector: $-x+5y = (-1+i\sqrt{10})x$; $5y = i\sqrt{10}x \rightarrow y = \frac{i\sqrt{10}}{5}x$, So $\begin{pmatrix}1\\\frac{i\sqrt{10}}{5}\end{pmatrix}$
7. Find general solution: $Y(t) = e^{\lambda_1 t}(Y_1)$

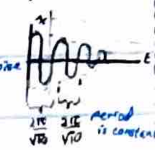
$Y(t) = e^{(-1+i\sqrt{10})t}\begin{pmatrix}1\\\frac{i\sqrt{10}}{5}\end{pmatrix} \rightarrow e^{-t}(\cos\sqrt{10}t + i\sin\sqrt{10}t)\begin{pmatrix}1\\\frac{i\sqrt{10}}{5}\end{pmatrix}$
$= \begin{pmatrix}e^{-t}\cos\sqrt{10}t\\-e^{-t}\frac{\sqrt{10}}{5}\sin\sqrt{10}t\end{pmatrix} + i\begin{pmatrix}e^{-t}\sin\sqrt{10}t\\e^{-t}\frac{\sqrt{10}}{5}\cos\sqrt{10}t\end{pmatrix}$

NEED REAL VALUED SOLUTIONS

$Y(t) = Y_{re}(t) + Y_{im}(t)$

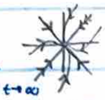
$Y(t) = k_1\begin{pmatrix}e^{-t}\cos\sqrt{10}t\\-e^{-t}\frac{\sqrt{10}}{5}\sin\sqrt{10}t\end{pmatrix} + k_2\begin{pmatrix}e^{-t}\sin\sqrt{10}t\\e^{-t}\frac{\sqrt{10}}{5}\cos\sqrt{10}t\end{pmatrix}$

8. Phase Plane



Cases for Linear Systems:

• 5 Common Cases:



1) $\lambda_1 < \lambda_2 < 0$ SINK
all solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$
all solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow -\infty$

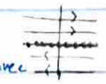
2) $\lambda_1 > \lambda_2 > 0$, SOURCE
all solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow \infty$
all solutions $\rightarrow \vec{0}$ as $t \rightarrow -\infty$

3) $\lambda_1 < 0 < \lambda_2$ SADDLE
(almost) all solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow \pm\infty$
↳ (not eigenvectors)

4) $\lambda = \alpha \pm \beta i$, $\alpha < 0$, SPIRAL SINK
all solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$
period: $2\pi/\beta$

5) $\lambda = \alpha \pm \beta i$, $\alpha > 0$ SPIRAL SOURCE
all solutions $\rightarrow \infty$
period: $2\pi/\beta$

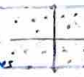
• 4 Weirdos:



11) Double zero eigenval, one eigenvec
solutions blow up linearly
Form: $\binom{k_1}{k_2} + t\binom{k_2}{0}$

12) Double negative eigenvals, all vecs eigenvecs
solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$
solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow -\infty$

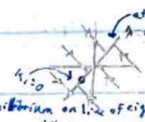
13) Double positive eigenvals, all vecs eigenvecs
solutions $\rightarrow \infty$ as $t \rightarrow \infty$
solutions $\rightarrow \vec{0}$ as $t \rightarrow -\infty$



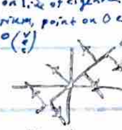
14) Double zero eigenvalue, all vectors eigenvecs
all points equilibria! $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

• 5 Dividing Cases:

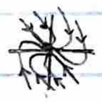
6) $\lambda = \alpha \pm \beta i$, $\alpha = 0$ CENTER
solutions oscillate w/ natural period $2\pi/\beta$



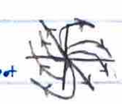
7) $\lambda_1 < \lambda_2 = 0$
solutions $\rightarrow$ equilibrium on line of eigenvec for $\lambda = 0$
if $k_1 = 0$, $Y(t)$ = equilibrium, point on 0 eigenvec
$Y(t) = k_1 e^{\lambda_1 t}\binom{v}{?} + k_2 e^{0}\binom{v}{?}$



8) $\lambda_1 > \lambda_2 = 0$
solutions $\rightarrow \infty$ as $t \rightarrow \infty$
(except line of equilibria)



9) $\lambda_1 = \lambda_2 < 0$ double negative root
solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$
solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow -\infty$



10) $\lambda_1 = \lambda_2 > 0$ double positive root
solutions $\rightarrow (\pm\infty, \pm\infty)$ as $t \rightarrow \infty$
solutions $\rightarrow \vec{0}$ as $t \rightarrow -\infty$

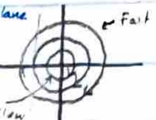
e.g. What happens when you get a center? (case 6)

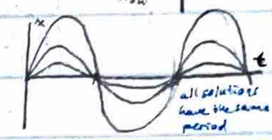
$\frac{dY}{dt} = \begin{bmatrix}0&3\\-3&0\end{bmatrix}Y$ $\det(A) = 0 - (-9) = 9$; $9 \neq 0$, so only $(0,0)$ is an equilibrium

1. Characteristic polynomial: $(0-\lambda)(0-\lambda) + 9 = 0 \rightarrow$ 2. eigenvals $\lambda = \pm 3i$
3. Class: $\alpha = 0$, so CENTER (solution oscillate w/ constant period & amplitude)
4. Period: $2\pi/\beta \rightarrow 2\pi/3$
5. Direction of spiral? $\begin{bmatrix}0&3\\-3&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix}$ solutions spiral clockwise
6. Eigenvec: $0x + 3y = 3ix$; $y = ix \rightarrow \binom{1}{i}$
7. General Solution: $e^{3it}\binom{1}{i} = (\cos 3t + i\sin 3t)\binom{1}{i}$
$= \binom{\cos 3t}{-\sin 3t} + i\binom{\sin 3t}{\cos 3t}$

So, $Y(t) = k_1\binom{\cos 3t}{-\sin 3t} + k_2\binom{\sin 3t}{\cos 3t}$

8. Phase plane

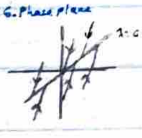




e.g. Procedure for case 7, 8

$\frac{dY}{dt} = \begin{bmatrix}2&-3\\4&-6\end{bmatrix}Y$ $\det(A) = -12 + 12 = 0$, line of equilibs

1. Characteristic Polynomial: $(2-\lambda)(-6-\lambda) + 12 = 0$; $\lambda^2 + 4\lambda = 0$
2. Eigenvals: $\lambda = 0, -4$
3. Eigenvecs: $\lambda_1 = -4$: $2x - 3y = -4x$, $y = 2x \rightarrow \begin{bmatrix}1\\2\end{bmatrix}$; $\lambda_2 = 0$: $2x - 3y = 0$, $3y = 2x \rightarrow \begin{bmatrix}3\\2\end{bmatrix}$
4. Class: one zero eigenvec, one negative eigenvalue, so line of equilibria and all other solutions approach this line.
5. General solution: $Y(t) = k_1 e^{-4t}\binom{1}{2} + k_2 e^{0}\binom{3}{2} \rightarrow Y(t) = k_1 e^{-4t}\binom{1}{2} + k_2\binom{3}{2}$
6. Phase plane

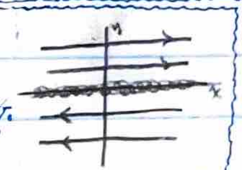


e.g. Procedure for case 11

$\frac{dY}{dt} = \begin{bmatrix}0&1\\0&0\end{bmatrix}Y$ $\det(A) = 0 + 0 = 0$ so line of equilibria

1. Char. Poly: $(0-\lambda)(0-\lambda) - 0 = 0$ 2. Eigenvals: $\lambda^2 = 0 \rightarrow \lambda = 0$
3. Class: double 0 eigenvalues
4. General solution: again use $Y(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1$, $V_1 = AV_0 - \lambda V_0$ ($\lambda = 0$)
Let $V_0 = \begin{bmatrix}k_1\\k_2\end{bmatrix}$, $V_1 = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} - 0\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_2\\0\end{bmatrix}$

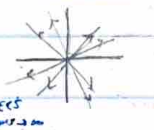
So, $Y(t) = e^{0t}\begin{bmatrix}k_1\\k_2\end{bmatrix} + te^{0t}\begin{bmatrix}k_2\\0\end{bmatrix} \rightarrow Y(t) = \begin{bmatrix}k_1\\k_2\end{bmatrix} + t\begin{bmatrix}k_2\\0\end{bmatrix}$



e.g. cases 12, 13, 14

$\frac{dY}{dt} = \begin{bmatrix}a&0\\0&d\end{bmatrix}Y$

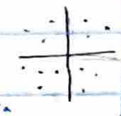
1. Char poly: $(a-\lambda)(d-\lambda) = 0$
2. Eigenvals: $\lambda = a, d$
3. ALL VECTORS ARE EIGENVECS
4. Since a, d positive, solutions $\rightarrow \infty$ as $t \rightarrow \infty$



if a, d negative, solutions would approach $\vec{0}$

$\frac{dY}{dt} = \begin{bmatrix}0&0\\0&0\end{bmatrix}Y$

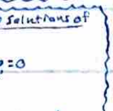
1. Char poly: $\lambda^2 = 0$
2. Eigenvals: 0, 0
3. NO EIGENVECTORS
4. All solutions are equilibria ↳ points



e.g. Procedure for cases 9, 10

$\frac{dY}{dt} = \begin{bmatrix}-2&1\\-1&-4\end{bmatrix}Y$ $\det(A) = 8 + 1 = 9$; $9 \neq 0$, only $\vec{0}$ equilibrium

1. Char Poly: $(-2-\lambda)(-4-\lambda) - (1)(-1) = 0$; $\lambda^2 + 6\lambda + 9 = 0$
2. Eigenvals: $(\lambda+3)(\lambda+3) = 0 \rightarrow \lambda = -3$
3. Class: double eigenval sink (solutions $\rightarrow \vec{0}$ as $t \rightarrow \infty$)
4. Eigenvector: $-2x + y = -3x$; $y = -x \rightarrow \begin{bmatrix}1\\-1\end{bmatrix}$ clockwise spiral
5. Direction of spiral: $\begin{bmatrix}-2&1\\-1&-4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}$
6. General Solution:
$Y(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1 \rightarrow V_0 = \binom{x_0}{y_0}$, $V_1 = AV_0 - \lambda V_0$
$Y(t) = e^{-3t}V_0 + te^{-3t}V_1$
7. IC: $Y_0 = \binom{2}{5} = V_0 \rightarrow V_1 = \begin{bmatrix}-2&1\\-1&-4\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} + 3\binom{2}{5}$
$V_1 = \begin{bmatrix}7\\-7\end{bmatrix}$
$Y(t) = e^{-3t}\begin{bmatrix}2\\5\end{bmatrix} + te^{-3t}\begin{bmatrix}7\\-7\end{bmatrix}$



Steps to Say Everything you can about solutions of $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$

1. Characteristic Polynomial: $\lambda^2 + p\lambda + q = 0$
2. eigenvals: $\frac{-p \pm \sqrt{p^2 - 4q}}{2}$
3. Classify: undamped, underdamped, overdamped, critically damped
4. Long term behavior
5. natural period: $2\pi/\beta$, $\beta = \frac{\sqrt{4q - p^2}}{2}$
6. Graph
7. General solution

Steps to find the general solution of a forced harmonic oscillator

1) Find the general solution of an unforced oscillator
$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$
2) Find one solution of forced equation
$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$
3) General solution is the sum of these
$$y(t) = y_g(t) + y_p(t)$$

Harmonic Oscillators

Formula: $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$ where $p = \frac{b}{m}$, $q = \frac{k}{m}$, b = damping constant, m = mass, k = spring constant

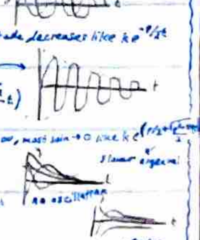
Characteristic Polynomial: $\lambda^2 + p\lambda + q = 0$

Eigenvalues: $\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

CASES:

- undamped: $p = 0$, $\lambda = \pm i\sqrt{q}$: solutions oscillate, amplitude constant & dependent on initial condition
Natural period $2\pi/\sqrt{q}$ $y(t) = k_1\cos(\sqrt{q}t) + k_2\sin(\sqrt{q}t)$
- underdamped: $p^2 - 4q < 0$
eigenvals: $\frac{-p}{2} \pm \frac{i\sqrt{4q - p^2}}{2}$, all solutions $\rightarrow 0$ as $t \rightarrow \infty$; oscillate, amplitude decreases like $ke^{-p/2t}$
Natural period: $2\pi / \frac{\sqrt{4q - p^2}}{2} = \frac{4\pi}{\sqrt{4q - p^2}}$ CONSTANT PERIOD →
General solution: $k_1 e^{(-p/2t)}\cos\left(\frac{\sqrt{4q - p^2}}{2}t\right) + k_2 e^{-p/2t}\sin\left(\frac{\sqrt{4q - p^2}}{2}t\right)$
- overdamped: $p^2 - 4q > 0$
eigenvals: $-\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$ (both always < 0) → all solutions $\rightarrow 0$ as $t \rightarrow \infty$, most soln $\rightarrow 0$ like $ke^{(-p/2 + \frac{\sqrt{p^2-4q}}{2})t}$ slower eigenval
General solution: $y(t) = \underbrace{k_1 e^{(-p/2 + \frac{\sqrt{p^2 - 4q}}{2})t}}_{\text{goes to 0 slow}} + \underbrace{k_2 e^{(-p/2 - \frac{\sqrt{p^2 - 4q}}{2})t}}_{\text{goes to 0 fast}}$ no oscillation
- critically damped: $p^2 - 4q = 0$
eigenval (double): $-p/2 < 0$, all solutions $\rightarrow 0$ as $t \rightarrow \infty$, no oscillation (most go like $k_2te^{-p/2t}$)
General solution: $y(t) = k_1 e^{-p/2t} + k_2 te^{-p/2t}$

TO SOLVE IVP:
- compute $y(0)$, $y'(0)$ to get a system of 2 equations w/ 2 unknowns



What to guess for forced responses:

- $g(t) = K \rightarrow y(t) = \alpha$
- $g(t) = Kt \rightarrow y(t) = \alpha t + \beta$
- $g(t) = Ke^{\omega t} \rightarrow y(t) = \alpha e^{\omega t}$; if ω is an eigenvec, guess $y(t) = \alpha te^{\omega t}$
- $g(t) = A\cos(\omega t)$ ↳ $y(t) = \alpha\cos\omega t + \beta\sin\omega t$

e.g. ~~overdamped~~ anti oscillator

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 12y = 0$$

1. charpoly: $\lambda^2 - \lambda - 12 = 0$
2. λ: $(\lambda-4)(\lambda+3)=0 \to \lambda = -3, 4$

~~3. classify: $\sqrt{1+48} > 0$ OVERDAMPED~~

~~4. solutions approach 0 as $t \to \infty$ ...~~

THIS ONE IS DUMB B/C P < 0

General solution: $Y(t) = k_1 e^{-3t} + k_2 e^{4t}$

---

e.g. overdamped oscillator

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$$

1. charpoly: $\lambda^2 + 2\lambda - 3 = 0$
2. eigenvals: $\lambda = \frac{-2 \pm \sqrt{16}}{2}$

JK AGAIN
I'M DUMB

---

e.g. overdamped harmonic oscillator

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0$$

1. char poly: $\lambda^2 + 8\lambda + 7 = 0$
2. eigenvals: $(\lambda+7)(\lambda+1) = 0$
   $\lambda = -7, -1$
3. Classify: $\sqrt{64-28} = \sqrt{36} > 0$, so oscillator is overdamped
4. Solutions $\to 0$ as $t \to \infty$; no oscillation, go to 0 like $k_1 e^{-t}$ (slower eigenval)
5. eigenvecs:
   $\begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} = \lambda \begin{pmatrix} y \\ v \end{pmatrix}$
   $\lambda = -1$: $v = -y \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   $\lambda = -7$: $v = -7y \to \begin{bmatrix} 1 \\ -7 \end{bmatrix}$
6. Natural period: no period b/c overdamped
7. General solution:
   $y(t) = k_1 e^{-t} + k_2 e^{-7t}$

graph

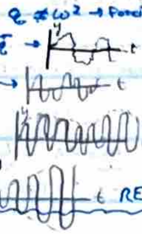
Forced Harmonic Oscillators (undamped)

Given: $\frac{d^2y}{dt^2} + qy = \cos(\omega t)$

$$y(t) = k_1 \cos(\sqrt{q}\,t) + k_2 \sin(\sqrt{q}\,t) + \frac{1}{q-\omega^2}\cos(\omega t)$$

$q \neq \omega^2 \to$ forcing period $\neq$ natural period

- $\omega >> \sqrt{q}$ $\to$ Forcing period: $2\pi/\omega$
- $\omega << \sqrt{q}$ $\to$
- $\omega \approx \sqrt{q}$ beats
- $\omega = \sqrt{q}$ RESONANCE. Amplitude $\to \infty$ as $t \to \infty$



Solving 2nd Order Diffeqs

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$$

$\lambda^2 + 2\lambda - 3 = 0$

$\lambda = \frac{-2 \pm \sqrt{4+12}}{2} = -3, +1$

$y(t) = k_1 e^{-3t} + 2k_2 e^{t}$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 29y = 0$$

$\lambda^2 - 4\lambda + 29 = 0$

$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 29}}{2} = 2 \pm 5i$

$y(t) = e^{(2+5i)t} = e^{2t}(\cos 5t + i \sin 5t)$

$y(t) = k_1 e^{2t}\cos 5t + k_2 e^{2t} \sin 5t$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0$$

$\lambda^2 + 4\lambda - 5 = 0$

$\lambda = \frac{-4 \pm \sqrt{16+20}}{2}$

$\lambda = 1, -5$

$y(t) = k_1 e^{t} + k_2 e^{-5t}$

$$\frac{d^2y}{dt^2} + 4\,dy/dt + 20y = 0$$

$\lambda^2 + 4\lambda + 20 = 0$

$\lambda = \frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm \frac{\sqrt{-64}}{2} = -2 \pm 4i$

$y(t) = e^{(-2+4i)t} = e^{-2t}(\cos 4t + i \sin 4t)$

$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t} \sin 4t$

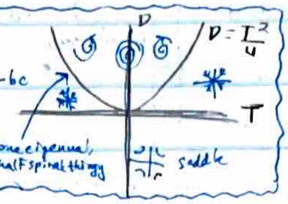
Bifurcation

given: $Y(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y$

charpoly: $\lambda^2 - (a+d)\lambda + ad - bc$

$\lambda^2 - T\lambda + D = 0$

$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$



Solving Forced 2nd Order equations

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{4t}$$

$\lambda = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} = 3, -2$

$y_g(t) = k_1 e^{3t} + k_2 e^{-2t}$

Guess: $y_p(t) = \alpha e^{4t}$

test:

$16\alpha e^{4t} - 4\alpha e^{4t} - 6\alpha e^{4t} = e^{4t}$

$6\alpha = 1 \to \alpha = 1/6$

So, $y(t) = k_1 e^{3t} + k_2 e^{-2t} + \frac{1}{6} e^{4t}$

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}$$

$\lambda = \frac{-7 \pm \sqrt{49-40}}{2} = -5, -2$

Guess: $y_p(t) = \alpha t e^{-2t}$

$y_p'(t) = \alpha e^{-2t} - 2\alpha t e^{-2t}$

$y_p''(t) = -2\alpha e^{-2t} - 2\alpha e^{-2t} + 4\alpha t e^{-2t}$

So, $4\alpha t e^{-2t} - 4\alpha e^{-2t} + 7(\alpha e^{-2t} - 2\alpha t e^{-2t}) + 10\alpha t e^{-2t} = e^{-2t}$

$4\alpha t - 4\alpha + 7\alpha - 14\alpha t + 10\alpha t = 1$

$3\alpha = 1 \to \alpha = 1/3$

$y(t) = k_1 e^{-5t} + k_2 e^{-2t} + \frac{1}{3} t e^{-2t}$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-t/2}$$

$\lambda = \frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm 4i$

$y_g(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t} \sin 4t$

Guess: $y_p(t) = \alpha e^{-t/2}$

Test: $\frac{1}{4}\alpha e^{-t/2} - 2\alpha e^{-t/2} + 20\alpha e^{-t/2} = e^{-t/2}$

$73\alpha = 4 \to \alpha = 4/73$

$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t + \frac{4}{73} e^{-t/2}$

$$\frac{d^2y}{dt^2} + 2y = -3$$

$\lambda = \frac{0 \pm \sqrt{-8}}{2} = \pm\sqrt{2}\,i$

$y_g(t) = k_1 \cos\sqrt{2}t + k_2 \sin\sqrt{2}t$

guess: $y_p(t) = \alpha \to$ test: $2\alpha = -3$, $\alpha = -3/2$

$y(t) = k_1 \cos\sqrt{2}t + k_2 \sin\sqrt{2}t - 3/2$

Sinusoidal Forcing

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = -4\cos 3t$$

$\lambda^2 + 6\lambda + 8 = 0$

$\lambda = \frac{-6 \pm \sqrt{36 - 4 \cdot 8}}{2} = -4, -2$

$y_g(t) = k_1 e^{-4t} + k_2 e^{-2t}$

$\omega = 3$

Guess: $y_p(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$

$y_p'(t) = -\alpha\omega\sin(\omega t) + \beta\omega\cos(\omega t)$

$y_p''(t) = -\alpha\omega^2\cos(\omega t) - \beta\omega^2\sin(\omega t)$

Plug in:

$-\alpha 9\cos(3t) - \beta 9\sin(3t) + 6(-\alpha 3\sin(3t) + \beta 3\cos(3t)) + 8(\alpha\cos(3t) + \beta\sin(3t)) = -4\cos(3t)$

etc

* if $\omega = \sqrt{p^2 - 4q}$, guess $\alpha t\cos(\omega t) + \beta t\sin(\omega t)$

BIFURCATION

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = \cos t$$

i) Nat: $\lambda^2 + 5\lambda + 3 = 0$ $\quad \lambda = \frac{-5 \pm \sqrt{25-12}}{2} = \frac{-5 \pm \sqrt{13}}{2}$ exponent. nat resp,

Forced: 2π C

ii) $-1 \pm \sqrt{1-12}$

Final Exam Example Problems

1.a) Say everything you can about solutions of the Harmonic Oscillator:

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+5y=0$$

• This is an unforced harmonic oscillator. The damping coefficient $4>0$, so all solutions $\to 0$ as $t\to\infty$ (exponentially fast).

• characteristic polynomial: $\lambda^2+4\lambda+5=0$

$$\lambda=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm\sqrt{-4}/2=-2\pm i$$

• $p^2-4q<0$, so this is an underdamped harmonic oscillator

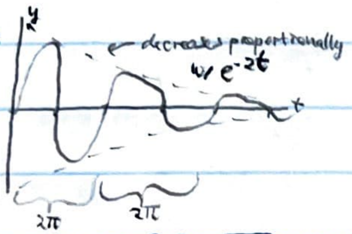
Solutions oscillate w/ decreasing amplitude (approaching 0) as $t\to\infty$
↳ decreases exponentially w ke^{-2t}

CONSTANT PERIOD

• Natural period: $2\pi/\beta=\frac{2\pi}{1}=2\pi$

• General solution: $y(t)=k_1e^{-2t}\cos t+k_2e^{-2t}\sin t$

• Graph:



1.b) Say everything you can about the solutions of:

$$\frac{d^2y}{dt^2}+\frac{1}{2}\frac{dy}{dt}+y=\cos(2t)$$

- This is a periodically forced harmonic oscillator
- damping coefficient >0, so natural response $\to 0$ as $t\to\infty$
- this means all solutions $\to$ forced response as $t\to\infty$
- Forced response is periodic w/ same period as forcing:

$$\frac{2\pi}{2}=\pi$$

- Unforced equation:
  - char poly: $\lambda^2+\frac{1}{2}\lambda+1=0$
  - eigenvals: $\frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-4}}{2}=-\frac{1}{4}\pm\frac{\sqrt{\frac{15}{4}}}{2}=\frac{\pm\sqrt{15}}{4}i-\frac{1}{4}$
  - General solution: $y_g(t)=k_1e^{-t/4}\cos(\frac{\sqrt{15}}{8}t)+k_2e^{-t/4}\sin(\frac{\sqrt{15}}{8}t)$
  - Natural period: $2\pi/\frac{\sqrt{15}}{4}=\frac{8\pi}{\sqrt{15}}\approx\pi$??

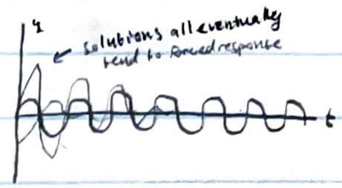
natural period < forced period, far from resonance so we expect the forced response to have a relatively small amplitude compared ~~forced response~~ to forcing amplitude 3)

• General solution of forced equation:
  - Find one solution of forced equation:

guess: $y_p(t)=\alpha\cos 2t+\beta\sin 2t$

$y_p'(t)=-2\alpha\sin 2t+2\beta\cos 2t$

$y_p''(t)=-4\alpha\cos 2t-4\beta\sin 2t$



test: $-4\alpha\cos 2t-4\beta\sin 2t+\frac{1}{2}(-2\alpha\sin 2t+2\beta\sin 2t)+\alpha\cos 2t+\beta\sin 2t=3\cos 2t$

$(-3\alpha+\beta)\cos 2t+(-3\beta-\alpha)\sin 2t=3\cos(2t)$

$3\alpha-\beta=3$ $\quad -3\beta-\alpha=0$, $\alpha=-3\beta$

$-3(-3\beta)+\beta=3$

$9\beta+\beta=3$

$\beta=3/10\to\alpha=-9/10$ $\quad y_p(t)=-9/10\cos 2t+3/10\sin 2t$

GENERAL SOLUTION OF FORCED EQUATION:

$y(t)=k_1e^{-t/4}\cos(\frac{\sqrt{15}}{4}t)+k_2e^{-t/4}\sin(\frac{\sqrt{15}}{4}t)-\frac{9}{10}\cos 2t+\frac{3}{10}\sin 2t$

The amplitude is $\sqrt{\frac{81}{100}+\frac{9}{100}}=\frac{\sqrt{90}}{10}$ (<1) (First two terms go to 0)

2. SAVING PROBLEM

You start saving money, depositing 5000 dollars/year and invest the money in an index stock fund which returns 4%/year compounded continuously.

a) Write a model for the growth of your investment

A = money in account $\quad A(0)=0$

t = time in years

$$\frac{dA}{dt}=0.04A+5000$$

b) Solve the IVP in part (a) to find amount saved by time t

$\frac{dA}{dt}=0.04A+5000$

$\frac{dA}{(0.04A+5000)}=dt$ $\quad u=0.04A+5000$

$\int\frac{dA}{(.04A+5000)}=\int dt$ $\quad du=0.04\,dA$, $dA=25\,du$

$25\int\frac{du}{|u|}=t+c$

$\ln|0.04A+5000|=(.04)(t+c)$ always positive

$|.04A+5000|=e^{.04t}e^{.04c}$ $\quad c_1=e^{.04c}$

$.04A+5000=c_1e^{.04t}$

$.04A=c_1e^{.04t}-5000$

$A=25c_1e^{.04t}-(25)(5000)$ $\quad c_2=c_1\cdot 25$

$A=c_2e^{.04t}-125,000$

$A(0)=0=c_2e^0-125,000$

$c_2=125,000$

$A(t)=125,000e^{.04t}-125,000$

c) Give a rough estimate of the amount of money you will have after 50 years

$A(50)=125,000e^{(.04)(50)}-125,000$ $\quad e\approx 2.7$

$=125,000e^2-125,000$ $\quad e^2\approx 7.3$

$A(50)=(125,000)(7.3)-125,000=(6.3)(125,000)$

$\approx 790,000$ dollars

d) Suppose you can now put in tax sheltered saving by 100 dollars a year. Modify your model from part A

$$\frac{dA}{dt}=.04A+5000+100t$$

3. Say everything you can about solutions of the system:

$$\frac{dY}{dt}=\begin{bmatrix}-2&-1\\3&-6\end{bmatrix}Y$$

1. Determinant: $\det(A)=(-2)(-6)-(-1)(3)=15\neq 0$, so only $\vec{0}$ is an equilibrium
2. Characteristic polynomial: $(-2-\lambda)(-6-\lambda)-(-1)(3)=0$

$\lambda^2+6\lambda+2\lambda+12+3=0$

$\lambda^2+8\lambda+15=0$

3. Eigenvalues: $(\lambda+5)(\lambda+3)=0$

$\lambda=-5,-3$

4. Classify: $\lambda_1<\lambda_2<0$, so this is a SINK. All solutions $\to\vec{0}$ as $t\to\infty$
Solutions $\to(\pm\infty,\pm\infty)$ as $t\to-\infty$ $(\neq(0,0))$

5. Eigenvectors: $\lambda=-5$

$\begin{bmatrix}-2&-1\\3&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-5\begin{bmatrix}x\\y\end{bmatrix}$

$-2x-y=-5x$

$y=3x\to\begin{bmatrix}1\\3\end{bmatrix}$

$\lambda=-3$

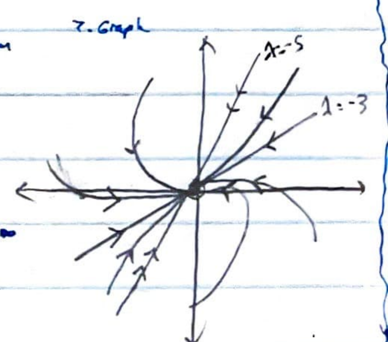
$\begin{bmatrix}-2&-1\\3&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-3\begin{bmatrix}x\\y\end{bmatrix}$

$-2x-y=-3x$

$y=x\to\begin{bmatrix}1\\1\end{bmatrix}$

6. General solution: $y(t)=k_1e^{-5t}\begin{bmatrix}1\\3\end{bmatrix}+k_2e^{-3t}\begin{bmatrix}1\\1\end{bmatrix}$

7. Graph



~~7. Graph~~

Money problem $\frac{dM}{dt}=0.05M-60,000$, $M(0)=M_0$

a) Compute an expression for when your money runs out

$\int\frac{dM}{(0.05M-60,000)}=\int dt$ $\quad u=(0.05M-60,000)$

$du=0.05\,dM\to dM=20\,du$

$20\int\frac{du}{|u|}=t+c$ ← always negative

$20\ln|0.05M-60,000|=t+c$

$-(0.05M-60,000)=e^{.05(t+c)}$ $\quad c_1=e^{.05c}$

$-0.05M=c_1e^{.05t}-60,000$

$M=-20c_1e^{.05t}-60,000$

BLAH w/ E