Exam 1 Study Guide

- parametric formula for dy/dx: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
- parametric formula for $\frac{d^2y}{dx^2}$: $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$
- length of a curve: $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)^2]}\, dt$

  a curve parameterized by $(x, f(x))$: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

- conversions: Polar and Cartesian Coordinates

  $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

- Symmetry tests for polar graphs

  1) Symmetry about the x-axis: if (r, θ) is on the graph, $(r, -\theta)$ is on the graph

  2) Symmetry about the y-axis: if (r, θ) is on the graph, $(-r, -\theta)$ is on the graph

  3) Symmetry about the origin: if (r, θ) lies on the graph, $(-r, \theta)$ is on the graph

- slope of $r = f(\theta)$ in the Cartesian plane: $\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$
- area of a fan-shaped region b/w the origin and the curve $r = f(\theta)$: $A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta$
- area of the region $0 \leq r_1\theta \leq r \leq r_2\theta$: $A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$
- Length of a Polar Curve: $L = \int_a^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
- distance b/w two points in space: $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- standard equation for a sphere: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$
- midpoint of a line segment: $m = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
- dot product: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$
- $\text{proj}_{\vec{v}}\vec{u}$ = the projection of $\vec{u}$ onto $\vec{v}$ = $\underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}_{\text{scalar component}} \cdot \frac{\vec{v}}{|\vec{v}|}$ • $\text{orth}_{\vec{v}}\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$
- work: $W = \vec{F} \cdot \vec{D}$
- cross product: $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta\, \hat{n} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$, $\vec{u} \parallel \vec{v}$ if $\vec{u} \times \vec{v} = 0$; $|\vec{v} \times \vec{u}|$ = area of parallelogram defined by $\vec{u}$ and $\vec{v}$
- area of a parallelepiped: $|(\vec{u} \times \vec{v}) \cdot w| = A$
- distance from a point S to a line through P parallel to $\vec{v}$: $d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$
- equation of a plane: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
- distance from a point to a plane: $d = \left|\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right|$
- arc length formula, revisited: $L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt = \int_a^b |\vec{v}|\, dt$
- unit tangent vector: $T = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{v}}{|\vec{v}|}$

Standard Equations of Quadric Surfaces

ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

elliptical cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

hyperboloid of two sheets: $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

hyperbolic paraboloid: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$, $c > 0$

Shit I definitely should have memorized by now

- Basic Derivatives:

• $\sin x \mapsto \cos x$ • $\sec x \mapsto \sec x \tan x$

• $\cos x \mapsto -\sin x$ • $\csc x \mapsto -\csc x \cot x$

• $\tan x \mapsto \sec^2 x$ • $\cot x \mapsto -\csc^2 x$

• $\sin^{-1} x \mapsto \frac{1}{\sqrt{1-x^2}}$ • $\sec^{-1} x \mapsto \frac{1}{|x|\sqrt{x^2-1}}$

• $\cos^{-1} x \mapsto \frac{-1}{\sqrt{1-x^2}}$ • $\csc^{-1} x \mapsto \frac{-1}{|x|\sqrt{x^2-1}}$

• $\tan^{-1} x \mapsto \frac{1}{1+x^2}$ • $\cot^{-1} x \mapsto \frac{-1}{1+x^2}$

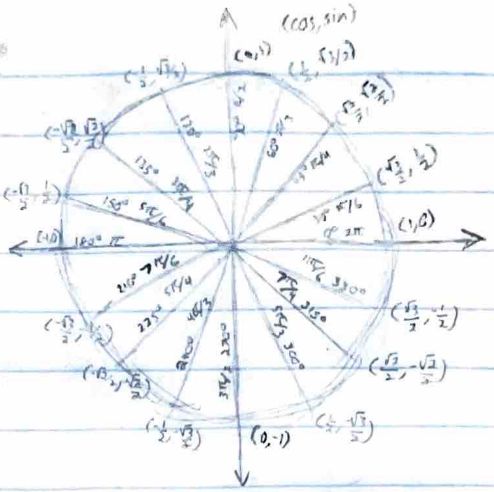
• $a^x \mapsto a^x \ln a$ • $\log_a x \mapsto \frac{1}{x \ln a}$

• $\ln x \mapsto \frac{1}{x}$

• Derivative Rules

- quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

- chain rule: $(f(g(x)))' = f'(g(x))\,g'(x)$



Trig identities:

$\sin^2 u + \cos^2 u = 1$ $\quad 1+\tan^2 u = \sec^2 u$ $\quad 1+\cot^2 u = \csc^2 u$

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$\sin 2u = 2 \sin u \cos u$

$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$

$\sin^2 u = \frac{1-\cos 2u}{2}$ $\quad \cos^2 u = \frac{1+\cos 2u}{2}$

- Basic Integrals:

• $\frac{1}{x} \mapsto \ln|x|$ • $\cos u \mapsto \sin u$

• $\frac{1}{ax+b} \mapsto \frac{1}{a}\ln|ax+b|$ • $\sin u \mapsto -\cos u$

• $x^n = \frac{1}{n+1} x^{n+1}$ • $\tan u \mapsto \ln|\sec u|$

16.9 #1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 23, 25, 27

#1 $\vec{F} = \langle \overset{M}{-y}, \overset{N}{x} \rangle$ $\quad \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$\int_0^{2\pi} (-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)$ $\quad R: x^2+y^2 \le a^2$ $\quad \frac{\partial N}{\partial x} = 1,\ \frac{\partial M}{\partial y} = -1$

$C: \vec{r} = \langle a\cos t, a\sin t \rangle$

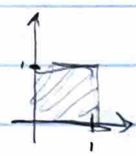
$a^2 \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi a^2$ $\quad \vec{r}' = \langle -a\sin t, a\cos t \rangle$

$\int_0^{2\pi}\int_0^a (1+1)r\,dr\,d\theta = \int_0^{2\pi}\int_0^a 2r\,dr\,d\theta = 2\pi a^2$

#5 $\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$\vec{F} = \langle \overset{M}{x-y}, \overset{N}{y-x} \rangle$ $\quad \frac{\partial N}{\partial x} = -1$ $\quad \frac{\partial M}{\partial y} = -1$

$\int_0^1\int_0^1 (-1 - -1)\,dx\,dy = 0$



#3 $\vec{F} = \langle \overset{M}{2x}, \overset{N}{3y} \rangle$ $\quad \vec{r} = \langle a\cos t, a\sin t \rangle$ $\quad \frac{\partial N}{\partial x} = 0$

$\vec{r}' = \langle -a\sin t, a\cos t \rangle$ $\quad \frac{\partial M}{\partial y} = 0$

$\int_C (2a\cos t)(-a\sin t) + (3a\sin t)(a\cos t)\,dt$

$-2a^2 \int_0^{2\pi} \cos t \sin t\,dt + 3a^2 \int \cos t \sin t\,dt = a^2 \int_0^{2\pi} \cos t \sin t\,dt = a^2 \int u\,du = a^2\left[\frac{\sin^2 t}{2}\right]_0^{2\pi} = 0$

$u = \sin t$
$du = \cos t\,dt$

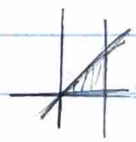
$\int_0^{2\pi}\int (0+0)\,dA = 0$

#7 $\vec{F} = \langle \overset{M}{y^2 - x^2}, \overset{N}{y^2 + x^2} \rangle$ $\quad \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x$

$\int_0^3\int_0^x (2x - 2y)\,dy\,dx$

$\int_0^3 (2xy - y^2)\Big|_0^x dx$

$\int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 3^2 = 9$



#9 $\vec{F} = \langle \overset{M}{xy + y^2}, \overset{N}{x - y} \rangle$ $\quad \frac{\partial N}{\partial x} = 1$ $\quad \frac{\partial M}{\partial y} = x + 2y$

$\int_0^1\int_{x^2}^{\sqrt{x}} (1 - x - 2y)\,dy\,dx = \int_0^1 (y - xy - y^2)\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x^{1/2} - x^{3/2} - x) - (x^2 - x^3 - x^4)\,dx$

$= \int_0^1 x^{1/2} - x^{3/2} - x - x^2 + x^3 + x^4\,dx$

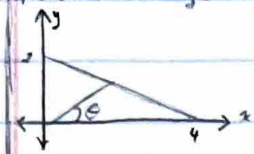
$= \frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5} - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\Big|_0^1$

$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$= \frac{1}{3} - \frac{1}{5} - \frac{1}{4} = \frac{20}{60} - \frac{12}{60} - \frac{15}{60} = \frac{-7}{60}$

Example Problems: Exam I

parameterize using θ as a parameter



$y = \frac{1}{2}x + 2$

$\tan\theta = \frac{y}{x} \rightarrow y = x\tan\theta$, y also $= \frac{1}{2}x + 2$

$x\tan\theta = \frac{1}{2}x + 2$

$2x\tan\theta = x + 4$

$x(2\tan\theta - 1) = 4$

So, $x = \frac{4}{2\tan\theta - 1}$

$y = \tan\theta\left(\frac{4}{2\tan\theta - 1}\right)$

Distance between two skew lines

$L_1 = \vec{r}_1(t) = \langle 1,0,2\rangle + t\langle 2,-1,-2\rangle$ ($\vec{v}_1$)

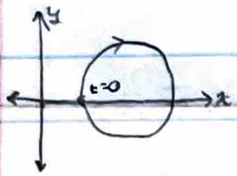
$L_2 = \vec{r}_2(t) = \langle 1,1,1\rangle + s\langle -1,1,1\rangle$ ($\vec{v}_2$)

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ -1 & 1 & 1 \end{vmatrix} = \langle 1,0,1\rangle$

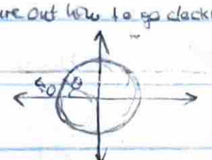
$\overrightarrow{L_1L_2} = \langle 1,0,2\rangle - \langle 1,1,1\rangle = \langle 0,1,-1\rangle$

$\text{proj}_{\overrightarrow{L_1L_2}}\vec{n} = \frac{\overrightarrow{L_1L_2} \cdot \vec{n}}{|\vec{n}|} = \left|\frac{-1}{\sqrt{2}}\right| = \sqrt{2}/2$

Parameterize $(x-2)^2 + y^2 = 1$



First, figure out how to go clockwise and start on the right for $x^2 + y^2 = 1$



$x(t) = \cos(\pi - \theta) = -\cos\theta$

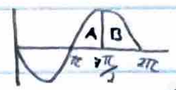
$y(t) = \sin(\pi - \theta) = \sin\theta$

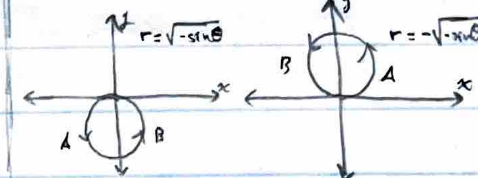
$x(t) = 2 - \cos t$

Now move it over: $y(t) = \sin t$

$r^2 = -\sin\theta$

plot: $-\sin\theta$





Find the graph of $r = 4\tan\theta\sec\theta$ in rectangular form

$r = 4\frac{\sin\theta}{\cos^2\theta}$

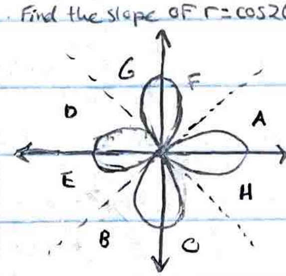
$r\cos^2\theta = 4\sin\theta$

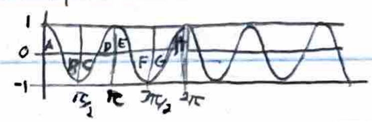
$r^2\cos^2\theta = 4r\sin\theta$

$x^2 = 4y \rightarrow y = \frac{x^2}{4}$

$0 \le \theta \le 2\pi$

Find the slope of $r = \cos 2\theta$ at different points on the curve

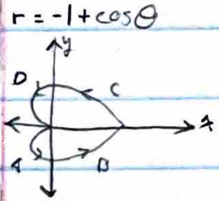




$x = r\cos\theta$ → function of r & θ $\quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$y = r\sin\theta$

$r = -1 + \cos\theta$



$\left.\frac{dy}{dx}\right|_{\theta = \pi/2} = -1$

$\left.\frac{dy}{dx}\right|_{\theta = -\pi/2} = 1$

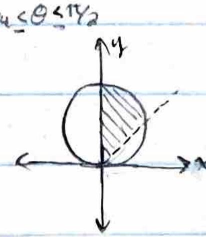
Area enclosed by $r = 2\sin\theta$, $\pi/4 \le \theta \le \pi/2$

$r = 2\sin\theta \rightarrow x^2 + (y-1)^2 = 1$

$A = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\sin\theta)^2 d\theta$

$= 2\int_{\pi/4}^{\pi/2} \sin^2\theta\, d\theta$

$= 2\int_{\pi/4}^{\pi/2} \frac{1 - \cos\theta}{2} d\theta = \pi/4 + 1/2$



$x' = \frac{dr}{d\theta}\cos\theta + r\left(\frac{d}{d\theta}\cos\theta\right)$

$y' = \frac{dr}{d\theta}\sin\theta + r\left(\frac{d}{d\theta}\sin\theta\right)$ } general

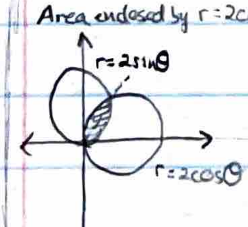
Since $r = \cos 2\theta$

$\frac{dx}{d\theta} = (-2\sin 2\theta)(\cos\theta) + \cos 2\theta(-\sin\theta)$

$\frac{dy}{d\theta} = (-2\sin 2\theta)\sin\theta + (\cos 2\theta)(\cos\theta)$

$\frac{dy}{dx} = 0$ at $\pi/2$

Area enclosed by $r = 2\cos\theta$ & $r = 2\sin\theta$



- Find bounds: $2\sin\theta = 2\cos\theta$ = common r
- $\theta = \pi/4$

$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2 d\theta = 2\int_0^{\pi/4} 2\sin^2\theta\, d\theta$

symmetry

$= 4\int_0^{\pi/4} \sin^2\theta\, d\theta$

$= 4\int_0^{\pi/4} \frac{1 - \cos\theta}{2} d\theta$

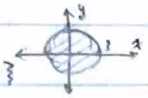
$\frac{\overrightarrow{L_1L_2} \cdot \vec{n}}{\vec{n}}$

L_1

Midterm 2: Example Problems

Find the domain of the following functions:

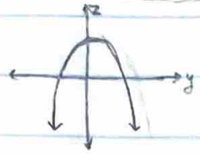
$z = f(x,y) = x^2 + y^2 \quad D = \mathbb{R}^2$



$z = f(x,y) = \sqrt{9 - x^2 - y^2} \quad D = \{(x,y) \mid x^2 + y^2 \le 9\}$

$z = \ln(x^2 + y) \quad D = \{(x,y) \mid y \le x^2\}$

$z = 6 - x - x^2 - 2y^2$ intersects plane $x = 1$



Find parametric equation for tangent line to this curve at $(1, 2, -4)$

when $x = 1$, $z = 6 - 1 - 1 - 2y^2 \rightarrow z = 4 - 2y^2$

$r(0) = (1, 2, -4)$, need direction vector $\vec{v}$, $\frac{\partial z}{\partial y} = -4y$, $\frac{\partial z}{\partial y}(1,2,-4) = -8$

$\therefore \vec{v} = \langle 0, 1, -8 \rangle$

$\vec{r}(t) = \langle 1, 2, -4 \rangle + t\langle 0, 1, -8 \rangle$

Find $\frac{\partial z}{\partial t}$ at $t = 0$, $z = x^2 y + 3xy^4$, $x(t) = \sin 2t$, $y(t) = \cos t$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$@ t = 0$, $x = 0$, $y = 1$ $\rightarrow \frac{\partial z}{\partial t}\Big|_{t=0} (0 + 3)(2\cos 0) + 0(\sin 0) = 6$

Find $\frac{\partial z}{\partial s}$ $\quad f(x,y) = e^x \sin y$, $x = st^2$, $y = s^2 t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= e^x \sin y\, t^2 + e^x \cos y\, 2st$$

Find the tangent line to the level curve $\frac{x^2}{4} + y^2 = 2$ at $(-2, 1)$

$\langle f_x, f_y \rangle \cdot \langle x - x_0, y - y_0 \rangle = \langle -1, 2 \rangle \cdot \langle x + 2, y - 1 \rangle = 0$

$\nabla f = \langle \frac{x}{2}, 2y \rangle = \langle -1, 2 \rangle \qquad -x - 2 + 2y - 2 = 0$

$2y - x = 4$

Find the equation of the plane tangent to $f(x,y) = 16 - 4x^2 - y^2$ at $(1, 1, 11)$

$\langle f_x, f_y, -1 \rangle \cdot \langle x - 1, y - 1, z - 11 \rangle = 0$

$f_x = -8x \quad f_y = -2y$

$= -8 \qquad = -2$

$\langle -8, -2, -1 \rangle \cdot \langle x - 1, y - 1, z - 11 \rangle = 0$

$-8x + 8 - 2y + 2 - z + 11$

$8x + 2y + z = 21$

For $x^2 + 2y^2 - 2x - 2z - 2 = 0$, are there any points at which the tangent plane is horizontal?

If the plane is horizontal, $\nabla f = \langle 0, 0, f_z \rangle$

$f_x = 2x - 2 = 0$, $f_y = 4y = 0$ $\Big\}$ $x = 1$, $y = 0$ $\qquad 1 - 2 - 2z - 2$, $2z = 3$, $z = 3/2$

Prove that the following limits do not exist

$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ approach along $y = 0$ $f(x,0) = \frac{x^2}{x^2} = 1$, $(x \neq 0)$, $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

approach along $x = 0$ $f(0,y) = \frac{-y^2}{y^2} = -1$, $\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

} different, so the limit does not exist

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ approach along $y = 0$ $f(x,0) = \frac{0}{x^2}$ $\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$

approach along $x = y$ $f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$, $\lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}$

} different, so the limit does not exist

$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$ this one has a limit! distance from $f(x,y)$ to $0 = z$ $\left\| \frac{3x^2 y}{x^2 + y^2} - 0 \right\| = \frac{x^2}{x^2 + y^2} \| 3y \|$, note that $\frac{x^2}{x^2 + y^2} < 1$ ALWAYS

$0 \le \frac{x^2}{x^2 + y^2} \| 3y \| \le \| 3y \|$, $\lim_{(x,y)\to(0,0)} 3\|y\| = 0$, so $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Find the following directional derivatives

$f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction $\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\nabla f = \langle 2x + y, x \rangle \rightarrow \langle 4, 1 \rangle$

$\langle 4, 1 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \frac{5\sqrt{2}}{2}$

$f(x,y) = x^2 + y^2$ in the direction of maximum slope at $(1,1)$

$\nabla f = \langle 2x, 2y \rangle \rightarrow \langle 2, 2 \rangle \quad |\nabla f| = \sqrt{2^2 + 2^2} = 2\sqrt{2} \quad \hat{u} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$\hat{D}_u f(1,1) = \langle 2, 2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$

Find a linear approximation of $f(x,y) = x\sqrt{y}$ at $(1,4)$ and use it to estimate the value of $f(x,y)$ at $f(1.1, 4.2)$

$f_x = \sqrt{y}$, $f_y = \frac{x}{2\sqrt{y}} \qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$

$f_x = 2 \quad f_y = \frac{1}{4} \qquad L(x,y) = 2 + 2(x - 1) + \frac{1}{4}(y - 4)$

$L(1.1, 4.2) = 2 + 2(.1) + \frac{1}{4}(.2)$

$= 2 + .2 + .05 = 2.25$

Find approximate change in $z = \ln(1 + x + y)$ from $(0,0)$ to $(-0.1, 0.03)$

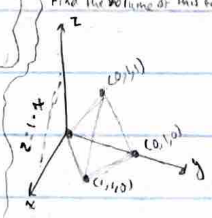
$z(0,0) = \ln 1 = 0$

$dz = f_x dx + f_y dy$

$dz = f_x dx + f_y dy$, $f_x = \frac{1}{1 + x + y} = 1$ $\quad f_y = \frac{1}{1 + x + y} = 1$, $\quad dz = 1(-0.1) + 1(0.03)$

$= -0.07$

Find the volume of this tetrahedron



Use xz plane as the floor

height = front plane − back plane ($y = 1$)

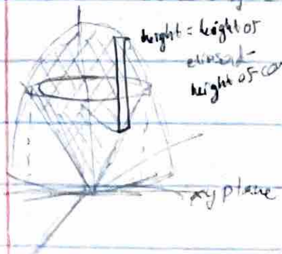
Find back plane: $\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle -1, 1, -1 \rangle$

$\langle -1, 1, -1 \rangle \cdot \langle 0, 0, 0 \rangle = 0$

$y = x + z$

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx$$

Find volume bounded below by $z = \sqrt{x^2 + y^2}$ and above by $x^2 + y^2 + z^2 = 8$



height = height of ellipsoid − height of cone

Floor: $2x^2 + 2y^2 = 8$

$x^2 + y^2 = 4$

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dz\, dy\, dx$$

Study Guide: 13.4-15.5

Sadie Allen

14.2 • a function $f(x,y)$ is continuous at (x_0,y_0) if

1) f is defined at (x_0,y_0)
2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

(a function is continuous if it is continuous at every point of its domain.

• Clairaut's Theorem: $f_{xy} = f_{yx}$

• Chain Rule: $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

• implicit differentiation: $\frac{dy}{dx} = -\frac{F_x}{F_y}$

• directional derivative: $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$ ← u = unit vector

- increases most rapidly in direction of $\vec{\nabla} f$
- decreases most rapidly in $-\vec{\nabla} f$ direction
- if $\hat{u} \perp \vec{\nabla} f$, $D_{\hat{u}} f = 0$

• tangent line to a level curve: $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$

• derivative along a path: $\frac{d}{dt} f(\vec{r}(t)) = \vec{\nabla} f(r(t)) \cdot r'(t)$

• tangent plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$: $f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$ $(\vec{\nabla} f \cdot \vec{v} = 0)$

• normal line to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$: $\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle f_x, f_y, f_z\rangle$

• plane tangent to a surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$: $f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = 0$ $(\vec{\nabla} f \cdot \vec{v} = 0)$

• estimating Δf in direction u: $df = D_u f \, ds$

• linearization of $f(x,y)$ at (x_0,y_0): $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

• total differential of f: $df = f_x\,dx + f_y\,dy$

• critical point: $f_x = f_y = 0$ OR one of f_x and f_y do not exist

• second derivative test:

i) maximum: $f_{xx} < 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$

ii) minimum: $f_{xx} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$

iii) saddle point: $f_{xx}f_{yy} - f_{xy}^2 < 0$

iv) inconclusive: $f_{xx}f_{yy} - f_{xy}^2 = 0$

• Lagrange Multipliers: $\vec{\nabla} f = \lambda \vec{\nabla} g$, $\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} g_2$

• Fubini's Theorem: $\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$

• area by integration: $A = \iint_R dA$

• average value of f over R: $\frac{1}{\text{Area of } R}\iint_R f\,dA$

• area in polar coordinates: $A = \iint_R r\,dr\,d\theta$

• volume w/ triple integrals: $V = \iiint_D dV$

Standard Equations of Quadric Surfaces:

ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

elliptical cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1$

hyperboloid of 2 sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1$

hyperbolic paraboloid: $\frac{y^2}{b^2} = \frac{z}{c} + \frac{x^2}{a^2}$, $c > 0$

• polar/cartesian conversions:

$\tan\theta = \frac{y}{x}$, $x = r\cos\theta$, $y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

MA225 Final Study Guide

- **Fubini's Theorem**: $\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$

relating rectangular & cylindrical coordinates

$x = r\cos\theta \quad r^2 = x^2+y^2$

$y = r\sin\theta \quad \tan\theta = y/x$

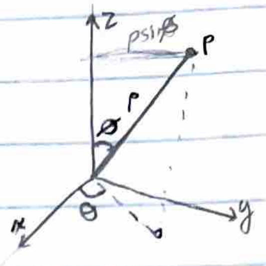
$z = z$

relating rectangular cylindrical & spherical coordinates

$r = \rho\sin\phi \quad x = r\cos\theta = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi \quad y = r\sin\theta = \rho\sin\phi\sin\theta$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$



- **Area by double integration**: $A = \iint_R 1\,dA$
- **Average value of f over R**: $\text{Avg} = \frac{1}{\text{Area of } R}\iint_R f\,dA$
- **Area in polar coordinates**: $A = \iint r\,dr\,d\theta$
- **volume by triple integration**: $V = \iiint_D dV$
- **Volume in cylindrical coordinates**: $V = \iiint dz\,r\,dr\,d\theta$
- **Volume in spherical coordinates**: $V = \iiint \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
- **Jacobian**: $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$
- **Substitution for double integrals**: $\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$
- **line integral**: $\int_C f(x,y,z)\,dt = \int_a^b f(g(t),h(t),k(t))\,|\vec{v}(t)|\,dt$
- **integral of a curve in a vector field $\vec{F}$**: $\int_C \vec{F}\cdot\vec{T} = \int_C \vec{F}\cdot d\vec{r} = \int \vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}\,dt$
- **Circulation**: if $\vec{r}(t)$ parameterizes a smooth curve C in the domain of a continuous velocity field $\vec{F}$, the flow along the curve from $A = \vec{r}(a)$ to $B = \vec{r}(b)$ is $\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds$. This is called a flow integral. If the curve starts and ends at the same point, so that $A = B$, the flow is called the circulation around the curve.
- **conservative vector field**: Let $\vec{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \vec{F}\cdot d\vec{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D and the field $\vec{F}$ is conservative on D.
- **potential function for F**: If $\vec{F}$ is a vector field defined on D and $\vec{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\vec{F}$
- **Fundamental Theorem of Line Integrals**: If C is a smooth curve joining A to B in the plane or in space and parameterized by $\vec{r}(t)$ and f is a differentiable function with a continuous gradient vector $\vec{F} = \nabla f$ on a domain D containing C, then

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

- **Loop property of conservative fields**: $\oint_C F\cdot dr = 0$ around every loop in D is equivalent to saying the field $\vec{F}$ is conservative on D.
- **Theorem: Conservative Fields are Gradient Fields**: Let $\vec{F} = \langle M, N, P\rangle$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\vec{F}$ is conservative iff $\vec{F}$ is a gradient field $\vec{F}$ for a differentiable function f
- **Component test for conservative fields**: Let $\vec{F} = \langle M,N,P\rangle$ be a field on an open, simply connected domain whose component functions have continuous first partial derivatives. $\vec{F}$ is conservative if: $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$
- **differential form**: any expression $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$. A differential form is exact on a domain D in space if $M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$ for some scalar function f in D
- **circulation density**: of vector field $\vec{F} = \langle M,N\rangle$ is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ (k-component of curl)
- **Green's Theorem (circulation form)**: $\oint_C F\cdot T\,ds = \oint_C M\,dx + N\,dy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$
- **Green's Theorem Area Formula**: Area of $R = \frac{1}{2}\oint_C x\,dy - y\,dx$

MA275 Practice Problems

Double integration over a simple region

$f(x,y)=12-x^2-2y^2$, $R=\{(x,y)\mid 1\le x\le 2,\ 0\le y\le 1\}$

$\int_1^2\int_0^1 (12-x^2-2y^2)\,dy\,dx$

$=\int_1^2\left[12y-x^2y-\frac{2}{3}y^3\right]_0^1 dy$

$=\int_1^2\left[12-x^2-\frac{2}{3}\right]dx=\left[12x-\frac{x^3}{3}-\frac{2}{3}x\right]_1^2$

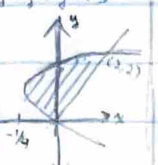
$=\left(24-\frac{8}{3}-\frac{4}{3}\right)-\left(12-\frac{1}{3}-\frac{2}{3}\right)$

$=(24-4)-(11)=10$

Double integration (irregular region)

$R=\{(x,y)\mid \text{bounded by } x=y^2-y \text{ \& } y=x\}$

$\int_0^2\int_{y^2-y}^{y} f(x,y)\,dx\,dy$



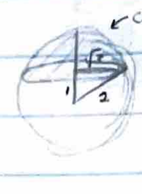
OR: $x=y^2-y=(y-\frac{1}{2})^2-\frac{1}{4}$

$y=\frac{1}{2}\pm\sqrt{x+1/4}$

$\int_{-1/4}^{0}\int_{1/2-\sqrt{x+1/4}}^{1/2+\sqrt{x+1/4}} f(x,y)\,dx\,dy+\int_0^2\int_x^{1/2+\sqrt{x+1/4}} f(x,y)\,dx\,dy$

Double integration: polar coordinates

Solid sphere rad = 2, plane one unit from center cuts it, find volume of cap



$\iint z\,dx\,dy=\iint\sqrt{2^2-x^2-y^2}\,dy\,dx$

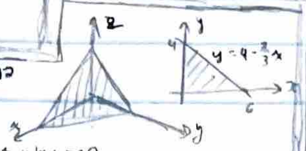
$x^2+y^2+z^2=2^2$

Solve for z to find height of dome

$z=\sqrt{4-x^2-y^2}-1$ polar

$\int_0^{2\pi}\int_0^{\sqrt{3}}\left(\sqrt{4-r^2}-1\right)r\,dr\,d\theta=5\pi/3$

Volume of tetrahedron defined by $2x+3y+6z=12$ in first quadrant.



$z=2-\frac{y}{2}-\frac{x}{3}$, "Floor looks like $y=4-\frac{2}{3}x$ when $z=0$

$\int_0^6\int_0^{4-\frac{2}{3}x}\int_0^{2-y/2-x/3} 1\,dz\,dy\,dx$

Find the volume of the region bounded below by $z=\sqrt{x^2+y^2}$ (cone) and above by $x^2+y^2+z^2=8$, $z\ge 0$

CARTESIAN $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\,dz\,dy\,dx$ SWITCH TO POLAR

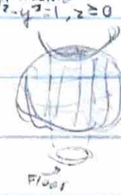
Polar $\int_0^{2\pi}\int_0^2\left(\sqrt{8-r^2}-\sqrt{r^2}\right)r\,dr\,d\theta=\frac{32\pi}{3}(\sqrt{2}-1)$

Area in the overlapping region between $x^2+y^2+z^2=19$ (sphere) and $z^2-x^2-y^2=1$ (hyperboloid), $z\ge 0$

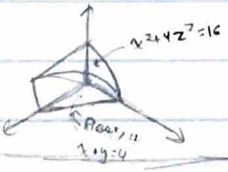
Find curve of intersection: $x^2+y^2+(x^2+y^2+1)=19$

$x^2+y^2=9$

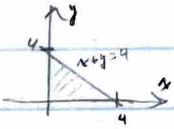
$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dz\,dy\,dx$ → then convert to polar



Find the area of this shape:



Floor: $(z=0,\ x+y=4)$

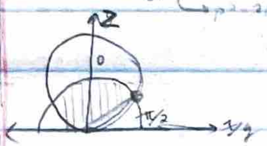


$\int_0^4\int_0^{4-y}\int_0^{\sqrt{4-\frac{x^2}{4}}} f(x,y,z)\,dz\,dx\,dy$

Volume bounded by $\rho=2\cos\phi$ and $\rho=1$, $z\ge 0$

$\rho^2=2\rho\cos\phi\to x^2+y^2+z^2=2z$

$x^2+y^2+(z-1)^2=1$

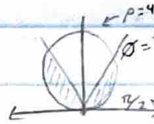


Where do they intersect:

$\rho=2\cos\phi=1\to\phi=\pi/3$

$V=\int_0^{2\pi}\int_0^{\pi/3}\int_0^1 1\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi} 1\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Volume outside $\phi=\pi/4$, inside $\rho=4\cos\phi$



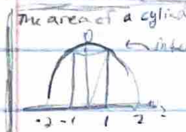
$V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{4\cos\phi} 1\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\int_0^{4\cos\phi}\rho^2\,d\rho$

$=2\pi\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\left.\frac{\rho^3}{3}\right|_0^{\rho=4\cos\phi}$

$=2\pi\int_{\pi/4}^{\pi/2}\frac{1}{3}\sin\phi(4\cos\phi)^3\,d\phi=\frac{2\pi}{3}(4)^3\left[\frac{-\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2}=\frac{8}{3}\pi$

The area of a cylinder w/ a spherical top

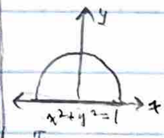


cylindrical coordinates: $V=\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 1r\,dz\,dr\,d\theta$ $\underset{\text{order}}{\overset{\text{different}}{\rightsquigarrow}}$ $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\,dr\,dz\,d\theta+\int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$

spherical coordinates: $V=\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{1/\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_0^{\pi/6}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Line integral

Take the integral of the function $2+x^2y$ over the curve shown.



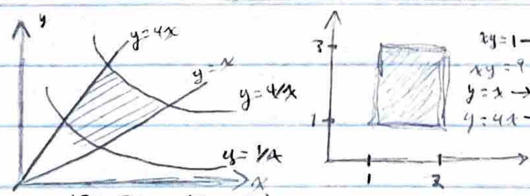
$\int_C f(x,y,z)\,ds=\int_a^b f(g(t),h(t),k(t))\,|\vec{v}(t)|\,dt$

$\vec{r}(t)=\langle\cos t,\sin t\rangle$ $0\le t\le\pi$, $\vec{r}'(t)=\langle-\sin t,\cos t\rangle$

$\int_0^{\pi}\left[2+\cos^2 t\sin t\right]dt=2\pi+2/3$

Jacobians

$\iint_R\left[\sqrt{\frac{y}{x}}+\sqrt{xy}\right]dx\,dy$



$xy=1\to u=1$

$xy=9\to u=3$

$y=x\to v=1$

$y=4x\to v=2$

let $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$ $u>0, v>0$

$x=\frac{u}{v}$, $y=uv$

$dx\,dy=|J|\,du\,dv=\left|2\frac{u}{v}\right|$

$\int_1^3\int_1^2(v+u)\left(\frac{2u}{v}\right)du\,dv$

$|J|=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{v}&\frac{-u}{v^2}\\ v&u\end{vmatrix}=\frac{2u}{v}$

Line integral over a vector field:

$\vec{F}=\langle x,-2,y\rangle$

$c=\vec{r}(t)=\langle 2t,3t,-t^2\rangle$ $-1\le t\le 1$

$\int_C\vec{F}\cdot d\vec{r}=\int_{-1}^{1}\langle 2t,t^2,3t\rangle\cdot\langle 2,3,-2t\rangle\,dt$

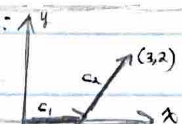
$=\int_{-1}^{1}(-3t^2+4t)\,dt=-2$

Line integral over a vector field:

$\vec{F}=\langle xy,x-y\rangle$

$C_1:\langle t,0\rangle$, $0\le t\le 2$

$dy/dt=0$



C_1: $\int_C\langle xy,x-y\rangle\cdot\langle dx,dy\rangle=\int_C xy\,dx+(x-y)\,dy$

$\int_C x(0)\,dx+(x-0)\,dy\overset{0}{}=0$

C_2: $y=2x-4$, $dy=2dx$, $2\le x\le 3$

$\int_2^3 x(2x-4)\,dx+(x-2x+4)(2dx)$

$=\int_2^3(2x^2-6x+8)\,dx=\frac{17}{3}$

Finding a potential function

$\vec{F}=\langle 2x^3+xy^2,\ 2y^3+x^2y\rangle$

$\frac{\partial\phi}{\partial x}=2x^3+xy^2$ $\xrightarrow[\text{to } x]{\text{integrate w/ respect}}$ $\phi(x,y)=\int(2x^3+xy^2)\,dx$

$=\frac{1}{2}x^4+\frac{y^2x^2}{2}+C(y)$

$\frac{\partial\phi}{\partial y}=2x^3+y^2y\longrightarrow\phi(x,y)=\int(2y^3+x^2y)\,dy$

$=\frac{y^4}{2}+\frac{x^2y^2}{2}+C(x)$

$\boxed{\phi(x,y)=\frac{x^4+y^2x^2+y^4}{2}}$

Testing conservative vector fields

$\vec{F}(x,y)=\langle\overset{f}{-y},\overset{g}{x+y}\rangle$ $\frac{\partial f}{\partial y}=-1$ $\frac{\partial g}{\partial x}=1$

NOT CONSERVATIVE

$F(x,y)=\langle 2x^3+xy^2,\ 2y^3+x^2y\rangle$

$\frac{\partial F}{\partial y}=2xy=\frac{\partial g}{\partial x}$

Conservative Vector Fields

$\vec{F}(x,y,z)=\langle x^3,2y,-z^3\rangle$

Now find $\phi(x,y,z)$

Check if conservative:

$\frac{\partial}{\partial y}(x^3)=\frac{\partial}{\partial x}(2y)=0$

$\frac{\partial}{\partial z}(x^3)=\frac{\partial}{\partial x}(-z^3)=0$ Yes!

$\frac{\partial}{\partial y}(-z^3)=\frac{\partial}{\partial z}(2y)=0$

$\frac{\partial\phi}{\partial y}=2y\to\int 2y\,dy=\phi(x,y,z)=y^2+c(x,z)$

$\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}\left[y^2+c(x,y)\right]=x^3\to c(x,z)=\frac{x^4}{4}+d(z)$

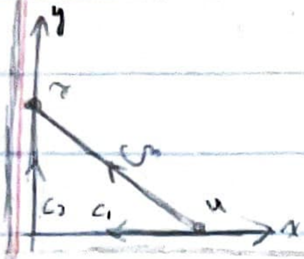
$\frac{\partial\phi}{\partial z}=\frac{\partial}{\partial z}\left[y^2+\frac{x^4}{4}+d(z)\right]=d'(z)=-z^3$

$d(z)=-\frac{z^4}{4}$

$\boxed{\phi(x,y,z)=y^2+\frac{x^4}{4}-\frac{z^4}{4}}$

Conservative Vector Fields

$\vec{F} = \langle x, y \rangle$, $\phi = \left(\frac{x^2+y^2}{2}\right)$



According to the Fundamental theorem of line integrals:

$\int_C \vec{F} \cdot d\vec{r} = f(b) - f(a)$

$= \left(\frac{2^2}{2}\right) - \left(\frac{4^2}{2}\right) = -6$

Conservative Vector Fields

$\vec{F} = \langle x, y \rangle$, $\phi = \left(\frac{x^2+y^2}{2}\right)$

$r(t) = \langle e^t \sin t, e^t \cos t \rangle$ new! $0 \le t \le \pi$

So... $\phi(b) - \phi(a)$

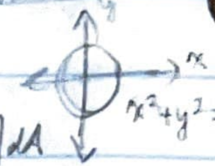
$\vec{r}_a = (0, 1)$

$\vec{r}_b = (0, -e^{\pi})$ $\phi(0, -e^{\pi}) - \phi(0, 1)$

$= \frac{1}{2}(e^{2\pi} - 1)$

Green's Theorem

$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$



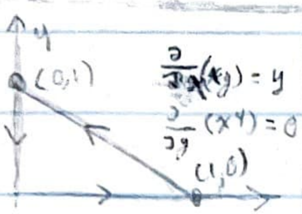
$x^2 + y^2 = 9$

$\iint \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$

$\iint_R (7-3)\,dx\,dy = \int_0^{2\pi}\int_0^{3} 4r\,dr\,d\theta = 36\pi$

Green's Theorem

$\oint \vec{F} \cdot d\vec{r}$, $\vec{F} = \langle x^4, xy \rangle$



$\frac{\partial}{\partial x}(xy) = y$

$\frac{\partial}{\partial y}(x^4) = 0$

$\oint \langle x^4, xy \rangle \cdot \langle dx, dy \rangle$

$\oint x^4\,dx + xy\,dy = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$

$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx = 1/6$

Green's Theorem in Reverse

Area of an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\vec{r}(t) = \langle a\cos t, b\sin t \rangle$

$r'(t) = \langle -a\sin t, b\cos t \rangle$

$R = \frac{1}{2}\oint_C x\,dy - y\,dx$

$R = \frac{1}{2}\int_C [(a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt]$

$= \frac{ab}{2}\int^{2\pi} \cos^2 t\,dt + \frac{ab}{2}\int_0^{2\pi} \sin^2 t$

Local Extrema

$f(x,y) = x^2 - y^2 - 2x + 4y + 6$, Find local maxima and minima

$f_x = 2x - 2$ $f_{xx} = 2$ $f_{xy} = 0$

$f_y = -2y + 4$ $f_{yy} = -2$

$2x - 2 = 0$ $x = 1$ crit point

$-2y + 4 = 0$ $y = 2$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$D = (2)(-2) - 0^2 = -4 < 0$, SADDLE POINT

$f(1,2) = 1^2 - 2^2 - 2 + 4 \cdot 2 + 6$

$= 9$

Saddle @ (1,2,9)

Local Extrema

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

$f_x = 2x + y + 3$ $f_{xx} = 2$ $f_{xy} = 1$

$f_y = x + 2y - 3$ $f_{yy} = 2$

Local extrema occur when:

$2x + y + 3 = 0$ $x + 2y - 3 = 0$

$y = -2x - 3$ $x + 2(-2x - 3) = 3$

$-3x - 6 = 3$

$x = -3$

$y = -2(-3) - 3$

$y = 6 - 3$

$= 6$

CRIT POINT: $x = -3$, $y = 6$

$D = f_{xx} f_{yy} - f_{xy}^2$ $f_{xx} > 0$

$D = (2)(2) - (1)^2 = 3 > 0$ LOCAL MIN

Absolute Extrema

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

region bounded by $x = 0$, $y = 2$, $y = 2x$

$f_x = 4x - 4$ $f_{xx} = 4$ $f_{xy} = 0$

$f_y = 2y - 4$ $f_{yy} = 2$

CHECK REGION:

Local extrema occurs when:

$4x - 4 = 0$ $2y - 4 = 0$

$x = 1$

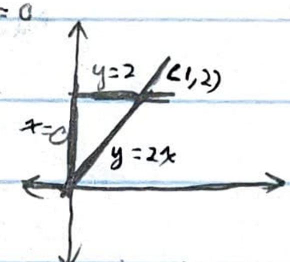
$y = 2$

$f(1,2) = -5$

$D = f_{xx} f_{yy} - f_{xy}^2$

$D = (4)(2) - 0 = 8 > 0$

$f_{xx} > 0$ LOCAL MIN



CHECK (OTHER) ENDPOINTS:

$f(0,0) = 1$

MAXIMUM: (0,0,1)

MINIMUM: (1,2,-5)

CHECK BORDERS:

$x = 0$: $f(0,y) = y^2 - 4y + 1$

$f'(0,y) = 2y - 4 \rightarrow y = 2$ is critical!

$f(0,2) = -3 \rightarrow (0, 2, -3)$

$y = 2$: $f(x,2) = 2x^2 - 4x - 3$

$f'(x,2) = 4x - 4 \rightarrow x = 1$, is critical!

$f(1,2) = -5 \rightarrow (1, 2, -5)$

$y = 2x$: $f(x,2x) = 6x^2 - 12x + 1$

$f'(x,2x) = 12x - 12 \rightarrow x = 1$ is crit

$f(1,2) = -5 \rightarrow (1, 2, -5)$